Value of the Stochastic Efficiency in Data Envelopment Analysis

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Abstract

This article examines the potential benefits of solving a stochastic DEA model over solving a deterministic DEA model. It demonstrates that wrong decisions could be made whenever a possible stochastic DEA problem is solved when the stochastic information is either unobserved or limited to a measure of central tendency. We propose two linear models: a semi-stochastic model where the inputs of the DMU of interest are treated as random while the inputs of the other DMUs are frozen at their expected values, and a stochastic model where the inputs of all of the DMUs are treated as random. These two models can be used with any empirical distribution in a Monte Carlo sampling approach. We also define the value of the stochastic efficiency (or semi-stochastic efficiency) and the expected value of the efficiency.

Keywords: Data Envelopment Analysis, Stochastic, Input-Output Analysis, Performance/Productivity

1. Introduction

Data envelopment analysis (DEA) is a technique that fundamentally measures the efficiency of homogenous entities of interest, which eventually allows identifying the best performers in the use of resources, pointing out where the potential gains may be made from possible improvements in efficiency, and helping the non-performers to achieve their potential. The DEA approach to technical efficiency measurement has its roots in the works of Koopmans (1951) and Farrell (1957). Later on, this approach was generalized to situations of multiple inputs and outputs and was reformulated as

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a mathematical programming problem by Charnes et al. (1978, 1979, 1981). Since then, many approaches have been developed to fulfill the requirement of real-life modeling, for instance, the additive models (Charnes and Cooper, 1984; Charnes et al., 1985), the multiplicative models (Charnes et al., 1982, 1983), the assurance region models (Thompson et al., 1986), and the cone ratio models (Charnes et al., 1989, 1990). Furthermore, Fried et al. (2008) edited a book on productive efficiency and productivity analysis, which encompassed over a decade of research from many known researchers in the field, such as Thanassoulis et al. (2008) and Fried et al. (2008). Notable surveys in DEA have been proposed by Charnes and Cooper (1984), Charnes et al. (1985), Banker et al. (1989), Seiford and Thrall (1990), Banker et al. (1994), Zhou et al. (2008), and Liu et al. (2013b,a).

Conventional DEA has been criticized for not allowing stochastic information to be incorporated in input and output data, which may, in turn, lead to the DEA efficiency measures to be sensitive to such information (Udhayakumar et al., 2011). In this context, in order to incorporate variations in inputs and outputs in the DEA analysis, Sengupta (1982), for example, generalized the Charnes-Cooper-Rhodes (Charnes et al., 1981) ratio model by defining the measure of efficiency of a DMU as the maximum of the sum of the expected ratio of weighted outputs to weighted inputs and a reliability function subject to several chance constraints. The concept of Stochastic Frontier Analysis was first proposed by Aigner et al. (1977) and Meeusen and van der Broeck (1977). Banker and Maindiratta (1992) proposed a similar semi-parametric DEA model, which has been recently improved by Kuosmanen (2008), Kuosmanen and Johnson (2010), Kuosmanen and Kortelainen (2012), and Kuosmanen et al. (2015). Based on the theory of chance-constrained programming, chance-constrained formulations of DEA were introduced by Sengupta (1989) and Desai and Schinnar (1987) in order to approach stochastic DEA. The chance-constrained programming approach was also adapted in the Land-Lovell-Thore (LLT) model (Land et al., 1993) and in the Olesen and Peterson (OP) model (Olesen and Petersen, 1995) to derive efficient frontiers that allow a part of the observed input-output combinations to be located on the wrong side of the frontiers. Furthermore, Olesen (2006) developed a merged model between the LLT model and the OP model that combines attractive features of each. The integration of reliability constraints in the OP chance-constrained model has been recently proposed by Wei et al. (2014). Additional aspects and applications of chance-constrained DEA models can be found in Olesen and Petersen (1995), Sueyoshi (2000), Cooper et al. (2002), Chen (2002), Cooper et al. (2004), and Talluri et al. (2006). Other contributions to the literature on stochastic DEA can be found in Banker (1986), Banker et al. (1987), Sengupta and Sfeir (1988), Banker (1993), Huang and Li (1996), Cooper et al. (1998), Li (1998), Sueyoshi (2000), Huang and Li (2001), Hall and Simar (2002), Ruggiero (2004), Simar (2007), and more recently in Branda and Kopa (2016). Recent applications on DEA dealing with uncertain data can be found in Udhayakumar et al. (2011), Charles (2014), Brandouy et al. (2015), Tsolas and Charles (2015) and Charles and Zavala (2017).

Olesen and Petersen (2015) have provided a review of Stochastic Data Envelopment Analysis in which they categorize the extensions of DEA into three directions: (i) extensions where deviations from the deterministic frontier are considered as random variables, (ii) extensions that integrate the random noise due to measurement errors, sample noise, or specification errors, and (iii) extensions that consider the Production Possibility Sets as stochastic. In this review, the *Management Science framework* is also opposed to the *econometric framework* in relation to the conclusions that could be drawn from the results.

It is to be noted that the above approaches suffer from a major drawback: they call for a continuous probability distribution. In other words, while maintaining the model size intact, nonlinearities increase the complexity of the problem. In this article, we propose to overcome this drawback by means of a two-stage stochastic program in a Monte Carlo sampling approach that can be used with any empirical distribution. Based on the *Management Science framework*, two linear models are proposed: a semi-stochastic model where the inputs of the DMU of interest are treated as random while the inputs of the other DMUs are frozen at their expected values, and a stochastic model where the inputs of all of the DMUs are treated as random. We also define the value of the stochastic efficiency (or semi-stochastic efficiency) and the expected value of the efficiency. For the sake of simplicity, the proposed models only consider random inputs, the generalization to random inputs and outputs being straightforward. To our knowledge, a two-stage stochastic programming approach has not been dealt in the DEA literature.

2. Deterministic DEA model

Let $J = \{DMU_j, j = 1, \dots, n\}$ be a set of decision making units (DMUs). We consider the problem of evaluating the relative efficiency of the DMUs while transforming a set M of inputs and a set P of outputs, where the inputs $m \in M$ for the DMU $j \in J$ are denoted by the non-negative matrix $\mathbf{X} = \left(x_{mj}\right)_{|M|\times|J|}$, and the outputs $n \in N$ for the DMU $j \in J$ are denoted by the non-negative matrix $\mathbf{Y} = \left(y_{nj}\right)_{|N|\times|J|}$. The input and output data being known, the objective is to evaluate the performance of one of the DMUs, i.e., the DMU of interest designated as DMU_0 and associated with the non-negative input and output vectors $\mathbf{x}_0 = [x_{m0}]$

and $\mathbf{y}_0 = [y_{n0}]$. The dual problem to a linear programming reformulation of the original fractional programming problem, in line with the Banker-Charnes-Cooper input-oriented DEA model (BCC) (Banker et al., 1984), is as follows:

$$(P1) \qquad \min \quad \phi \tag{1}$$

$$\text{s.t.} \qquad \sum_{j\in J} y_{pj}\lambda_j \geq y_{p0}, \qquad \forall p \in P \tag{2}$$

$$\sum_{j\in J} x_{mj}\lambda_j \leq \phi x_{m0}, \qquad \forall m \in M \tag{3}$$

$$\sum_{j\in J} \lambda_j = 1, \tag{4}$$

$$\lambda_j \geq 0, \qquad \forall j \in J \tag{5}$$

where λ_j coefficients are called *structural variables*, and ϕ is the so-called efficiency score, which lies in the closed interval $[0, 1]$. If ϕ is less than one, a proportional reduction of all of the inputs is needed in order to reach the efficient frontier. This reduction is given by $(1-\phi)\mathbf{x}_0$ which means that the projected unit given by $(\mathbf{x}_0, \mathbf{y}_0)$ is weakly efficient in DEA terminology. No further radial reduction of all of the inputs is possible given the present amount of outputs. It is possible that, in order to be DEA-efficient, further individual reductions or increases in some inputs and outputs are required. To evaluate these mix-inefficiencies one needs to resort to an extra multifaceted BCC model, in which a *non-Archimedean* element has to be introduced in the model. The said *non-Archimedean* element guarantees that the sum of the slacks is always maximized without altering the value of the efficiency scores at the optimum. This condition, introduced by Charnes et al. (1979), is needed in order to ensure that the projected unit belongs to the efficient frontier. If one wishes to discriminate between efficient and weakly efficient DMUs, then slack variables are introduced. In this case, the mathematical formulation of DEA, in line with the BCC model, is defined as follows:

$$(P2) \qquad \min \quad \phi - \epsilon\left(\sum_{m\in M} s_m^- + \sum_{p\in P} s_p^+\right) \tag{6}$$

$$\text{s.t.} \qquad \sum_{j\in J} y_{pj}\lambda_j - s_p^+ = y_{p0}, \qquad \forall p \in P \tag{7}$$

$$\sum_{j \in J} x_{mj}\lambda_j + s_m^- = \phi x_{m0}, \qquad \forall m \in M \tag{8}$$

$$\sum_{j \in J} \lambda_j = 1, \tag{9}$$

$$s_m^- \geq 0, \qquad \forall m \in M \tag{10}$$

$$s_p^+ \geq 0, \qquad \forall p \in P \tag{11}$$

$$\lambda_j \geq 0, \qquad \forall j \in J \tag{12}$$

where s_m^- and s_p^+ are the input and output slack variables, respectively, and ϵ is a non-Archimedean infinitesimal number. The optimization of the model (P2) is a two-stage process: the first stage optimizes model (P1), i.e., ϕ is minimized so as to obtain the *purely technical efficiency*, while the sum of the slack variables is maximized to yield a measure of *mix-inefficiency* in the second stage. The resulting objective function lies in the closed interval $[0, 1]$ and the DMU of interest is said to be efficient when it takes the value one.

Definition 1. *The DMU of interest (DMU_0) is technically strongly efficient if and only if ϕ^* is equal to one and all of the slack variables are equal to zero.*

Definition 2. *The DMU of interest (DMU_0) is technically weakly efficient if and only if ϕ^* is equal to one and at least one of the slack variables is non-zero.*

3. DEA model with stochastic variables

Using the above model we can extend our characterization to stochastic input variables. Let K be the set of stochastic inputs, Ω be the finite set of all of the realizations, and the non-negative matrix $\mathbf{X}(\omega) = \big(x_{kj}(\omega)\big)_{|K|\times|J|}$ where $x_{kj}(\omega)$ denotes the stochastic input $k \in K$ for DMU_j (here, the functional forms $\mathbf{X}(\omega)$ and $x_{kj}(\omega)$ are used to show the explicit dependence on ω, not to denote their realizations); we consider the problem of evaluating the relative efficiency of the n DMUs with $|M|$ deterministic inputs and $|K|$ stochastic inputs used to produce $|P|$ outputs. Except for the stochastic inputs, all other inputs and outputs are known. The objective is to evaluate the performance of DMU_0, the DMU of interest, associated with the non-negative input and output vectors $\mathbf{x}_0$, $\mathbf{x}_0(\omega) = \big[x_{k0}(\omega)\big]$, and $\mathbf{y}_0$. The mathematical formulation of DEA in the presence of stochastic inputs, in line with the BCC Banker et al. (1984) input-oriented model, is as follows:

$$(P3) \qquad \min \quad \phi \tag{13}$$

$$\text{s.t.} \qquad \sum_{j\in J} y_{pj}\lambda_j \geq y_{p0}, \qquad \forall p \in P \tag{14}$$

$$\sum_{j\in J} x_{mj}\lambda_j \leq \phi x_{m0}, \qquad \forall m \in M \tag{15}$$

$$\sum_{j\in J} x_{kj}(\omega), \lambda_j \leq x_{k0}(\omega) \qquad \forall k \in K \tag{16}$$

$$\sum_{j\in J} \lambda_j = 1, \tag{17}$$

$$\lambda_j \geq 0, \qquad \forall j \in J. \tag{18}$$

This is the *first stage program*, by which the first stage variables λ_j, defined for an estimated value of the stochastic inputs, are determined here and now before their actual value $x_{kj}(\omega)$ becomes known.

If $\delta_k(\omega) = x_{k0}(\omega) - \sum_{j\in J} x_{kj}(\omega)\lambda_j$ denotes the second stage decision variables for each stochastic input $k \in K$ and π_k is a positive penalty associated with the discrepancy between $x_{kj}(\omega)$ and $\sum_{j\in J} x_{kj}(\omega)\lambda_j$, then we solve the following program:

$$(P4) \qquad \min \quad \sum_{k\in K} \pi_k|\delta_k| \tag{19}$$

$$\text{s.t.} \qquad \delta_k(\omega) = x_{k0}(\omega) - \sum_{j\in J} x_{kj}(\omega)\lambda_j, \qquad \forall k \in K \tag{20}$$

where $x_{kj}(\omega)$, $x_{k0}(\omega)$, and λ_j are known, and so are y_{pj}, x_{mj}, and π_k. The problem of determining a recourse vector δ most economically, once the actual value of the stochastic inputs $x_{kj}(\omega)$ becomes known, is called the *second stage program.* Three cases arise:

Expected value of the efficiency (EVE). A natural temptation is to solve a simpler problem by allowing the random inputs of every DMU to freeze at their respective expected values, i.e.:

$$\delta_k = E\left[x_{k0}(\omega)\right] - \sum_{j\in J} E\left[x_{kj}(\omega)\right]\lambda_j, \qquad k \in K \tag{21}$$

where $E[\bullet]$ denotes the expected value among all of the realizations $\omega \in \Omega$. The corresponding problem is known as the *expected value problem* or the *mean value problem.*

Semi-stochastic efficiency (SSE). Another option would be to allow the random inputs of the DMU of interest to vary and to freeze the random inputs of all of the other DMUs at their mean points, i.e.:

$$\delta_k(\omega) = x_{k0}(\omega)(1-\lambda_0) - \sum_{j \in J \setminus \{0\}} E\left[x_{kj}(\omega)\right]\lambda_j, \qquad k \in K. \tag{22}$$

Stochastic efficiency (SE). A third option is to allow the random inputs of all of the DMUs to vary, i.e.:

$$\delta_k(\omega) = x_{k0}(\omega) - \sum_{j \in J} x_{kj}(\omega)\lambda_j, \qquad k \in K. \tag{23}$$

Let us derive the *semi-stochastic efficiency* model; the *expected value of the efficiency* and the *stochastic efficiency* models can be derived in line with the following arguments. Program (P4) is equivalent to finding $\delta_k^+(\omega)$ and $\delta_k^-(\omega)$ which solves:

$$(P5) \qquad \min \quad E\left[\sum_{k \in K} \pi_k \left(\delta_k^+(\omega) + \delta_k^-(\omega)\right)\right] \tag{24}$$

$$\text{s.t.} \qquad \delta_k^+(\omega) - \delta_k^-(\omega) = x_{k0}(\omega)(1-\lambda_0) - \sum_{j \in J} E\left[x_{kj}(\omega)\right]\lambda_j, \qquad \forall k \in K, \forall \omega \in \Omega \tag{25}$$

$$\delta_k^+(\omega) \geq 0, \qquad \forall k \in K \tag{26}$$

$$\delta_k^-(\omega) \geq 0, \qquad \forall k \in K \tag{27}$$

where $x_{kj}(\omega)$, $x_{k0}(\omega)$, and λ_j are known. The slack variable $\delta_k^-(\omega)$ represents the overachievement of the k^{th} stochastic constraint whereas the slack variable $\delta_k^+(\omega)$ represents its underachievement. Combining (P3) and (P4), the two stage stochastic program is reduced to the following convex program:

$$(P6) \qquad \min \quad \phi + E\left[\sum_{k \in K} \pi_k \left| x_{k0}(\omega)(1-\lambda_0) - \sum_{j \in J \setminus \{0\}} E\left[x_{kj}(\omega)\right]\lambda_j \right|\right] \tag{28}$$

$$\text{s.t.} \qquad \sum_{j \in J} y_{pj}\lambda_j \geq y_{p0}, \qquad \forall p \in P \tag{29}$$

$$\sum_{j\in J} x_{mj}\lambda_j \le \phi x_{m0}, \qquad \forall m \in M \tag{30}$$

$$\sum_{j\in J} \lambda_j = 1, \tag{31}$$

$$\lambda_j \ge 0, \qquad \forall j \in J. \tag{32}$$

In (P5), the penalties π_k are associated with the slack variables $\delta_k^+(\omega)$ and $\delta_k^-(\omega)$. It is to be noted that it does not make sense to use the same penalty for positive and negative violations. Hence, we can associate different penalties π_k^+ and π_k^- with the slack variables $\delta_k^+(\omega)$ and $\delta_k^-(\omega)$, respectively. Then, (P6) can be rewritten as follows:

$$(P7) \qquad \min \quad \phi + E\left[\sum_{k\in K}\left(\pi_k^+\delta_k^+(\omega) + \pi_k^-\delta_k^-(\omega)\right)\right] \tag{33}$$

$$\text{s.t.} \qquad \sum_{j\in J} y_{pj}\lambda_j \ge y_{p0}, \qquad \forall p \in P \tag{34}$$

$$\sum_{j\in J} x_{mj}\lambda_j \le \phi x_{m0}, \qquad \forall m \in M \tag{35}$$

$$\delta_k^+(\omega) - \delta_k^-(\omega) = x_{k0}(\omega)(1-\lambda_0) - \sum_{j\in J\setminus\{0\}} E\left[x_{kj}(\omega)\right]\lambda_j, \qquad \forall k \in K, \forall \omega \in \Omega \tag{36}$$

$$\sum_{j\in J} \lambda_j = 1, \tag{37}$$

$$\lambda_j \ge 0, \qquad \forall j \in J \tag{38}$$

$$\delta_k^+(\omega) \ge 0, \qquad \forall k \in K, \forall \omega \in \Omega \tag{39}$$

$$\delta_k^-(\omega) \ge 0, \qquad \forall k \in K, \forall \omega \in \Omega. \tag{40}$$

In view of (P2), the above model can be rewritten as:

$$(P8) \qquad SSE = \min \quad \phi + E\left[\sum_{k\in K}\left(\pi_k^+\delta_k^+(\omega) + \pi_k^-\delta_k^-(\omega)\right)\right] - \epsilon\left(\sum_{m\in M} s_m^- + \sum_{p\in P} s_p^+\right) \tag{41}$$

$$\text{s.t.} \qquad \sum_{j\in J} y_{pj}\lambda_j - s_p^+ = y_{p0}, \qquad \forall p \in P \tag{42}$$

$$\sum_{j\in J} x_{mj}\lambda_j + s_m^- = \phi x_{m0}, \qquad \forall m \in M \tag{43}$$

$$\delta_k^+(\omega) - \delta_k^-(\omega) = x_{k0}(\omega)(1-\lambda_0) - \sum_{j \in J \setminus \{0\}} E\left[x_{kj}(\omega)\right]\lambda_j, \qquad \forall k \in K, \forall \omega \in \Omega \quad (44)$$

$$\sum_{j \in J} \lambda_j = 1, \quad (45)$$

$$\lambda_j \geq 0, \qquad \forall j \in J, \quad (46)$$

$$\delta_k^+(\omega) \geq 0, \qquad \forall k \in K, \forall \omega \in \Omega \quad (47)$$

$$\delta_k^-(\omega) \geq 0, \qquad \forall k \in K, \forall \omega \in \Omega \quad (48)$$

$$s_m^- \geq 0, \qquad \forall m \in M \quad (49)$$

$$s_p^+ \geq 0, \qquad \forall p \in P. \quad (50)$$

It is worth noting that the objective function lies in the closed interval $[0, 1]$ and is equal to one if and only if the DMU of interest is semi-stochastically efficient.

In a similar way, we obtain the following *stochastic efficiency* model:

$$(P9) \qquad SE = \min \quad \phi + E\left[\sum_{k \in K} \left(\pi_k^+ \delta_k^+(\omega) + \pi_k^- \delta_k^-(\omega)\right)\right] - \epsilon\left(\sum_{m \in M} s_m^- + \sum_{p \in P} s_p^+\right) \quad (51)$$

$$\text{s.t.} \qquad \sum_{j \in J} y_{pj}\lambda_j - s_p^+ = y_{p0}, \qquad \forall p \in P \quad (52)$$

$$\sum_{j \in J} x_{mj}\lambda_j + s_m^- = \phi x_{m0}, \qquad \forall m \in M \quad (53)$$

$$\delta_k^+(\omega) - \delta_k^-(\omega) = x_{k0}(\omega) - \sum_{j \in J} x_{kj}(\omega)\lambda_j, \qquad \forall k \in K, \forall \omega \in \Omega \quad (54)$$

$$\sum_{j \in J} \lambda_j = 1, \quad (55)$$

$$\lambda_j \geq 0, \qquad \forall j \in J \quad (56)$$

$$\delta_k^+(\omega) \geq 0, \qquad \forall k \in K, \forall \omega \in \Omega \quad (57)$$

$$\delta_k^-(\omega) \geq 0, \qquad \forall k \in K, \forall \omega \in \Omega \quad (58)$$

$$s_m^- \geq 0, \qquad \forall m \in M \quad (59)$$

$$s_p^+ \geq 0, \qquad \forall p \in P \quad (60)$$

and the *expected value of the efficiency* model:

$$(P10) \qquad EVE = \min \quad \phi + \sum_{k \in K} \left(\pi_k^+ \delta_k^+ + \pi_k^- \delta_k^-\right) - \epsilon\left(\sum_{m \in M} s_m^- + \sum_{p \in P} s_p^+\right) \quad (61)$$

$$
\begin{aligned}
\text{s.t.} \qquad \sum_{j \in J} y_{pj}\lambda_j - s_p^+ &= y_{p0}, && \forall p \in P && (62)\\
\sum_{j \in J} x_{mj}\lambda_j + s_m^- &= \phi x_{m0}, && \forall m \in M && (63)\\
\delta_k^+ - \delta_k^- = E\left[x_{k0}(\omega)\right] - \sum_{j \in J} E\left[x_{kj}(\omega)\right]\lambda_j&, && \forall k \in K && (64)\\
\sum_{j \in J} \lambda_j &= 1, && && (65)\\
\lambda_j &\geq 0, && \forall j \in J && (66)\\
\delta_k^+ &\geq 0, && \forall k \in K && (67)\\
\delta_k^- &\geq 0, && \forall k \in K && (68)\\
s_m^- &\geq 0, && \forall m \in M && (69)\\
s_p^+ &\geq 0, && \forall p \in P. && (70)
\end{aligned}
$$

Definition 3. *The DMU of interest, DMU_0, is said to be technically strongly stochastic efficient if and only if the following conditions are all satisfied when solving (P9):*

i) $\phi^* = 1$,

ii) $\delta_k^{*+}(\omega) = \delta_k^{*-}(\omega) = 0, \qquad \forall k \in K, \forall \omega \in \Omega,$

iii) $s_p^{*+} = s_m^{*-} = 0, \qquad \forall (p, m) \in P \times M.$

It is said to be technically strongly semi-stochastic efficient when the same conditions are all satisfied when solving (P8), and technically strongly efficient when (i) and (iii) along with $\delta_k^{*+} = \delta_k^{*-} = 0$ are satisfied for any $k \in K$ when solving (P10). The DMU of interest is said to be technically weakly semi-stochastic efficient and technically weakly efficient, respectively, for problems P(8) and P(10), when (i) is satisfied and at least one of the conditions (ii) and (iii) is not satisfied when solving P(8), and when (i) is satisfied and at least one of the conditions $\delta_k^{*+} = \delta_k^{*-} = 0$ and (iii) is not satisfied for any $(k, p, m) \in K \times P \times M$ when solving P(10).

Proposition. $SE \geq EVE$.

Proof. Let $X' = \left\{\left(x_{kj}(\omega)\right) \mid k \in K, j \in J \setminus \{0\}, \omega \in \Omega\right\}$ be the set of coefficients in the constraints (54) corresponding to all of the DMUs except the DMU of interest, and let $X'' = \left\{\left[x_{k0}(\omega)\right] \mid k \in K, \omega \in \Omega\right\}$ be the set of coefficients corresponding to the DMU of interest. If we express SSE as a function $Z\left(X', X''\right)$, we can rewrite

EVE as $Z\left(E\left[X'\right], E\left[X''\right]\right)$. As Z is convex in X' and X'' (see Proposition 4.2 in Birge and Louveaux (2011) for a proof of the convexity of SE), by Jensen's inequality, we have:

$$Z\left(X', X''\right) \geq Z\left(E\left[X'\right], E\left[X''\right]\right), \tag{71}$$

that is:

$$SE \geq EVE. \tag{72}$$

□

Proposition. *$SSE \geq EVE$.*

Proof. Let $X' = \left\{E\left[\left(x_{kj}(\omega)\right)\right] \mid k \in K, j \in J \setminus \{0\}\right\}$ be the set of coefficients in the constraints (44) corresponding to all of the DMUs except the DMU of interest, and let $X'' = \left\{\left[x_{k0}(\omega)\right] \mid k \in K, \omega \in \Omega\right\}$ be the set of coefficients corresponding to the DMU of interest. If we express SSE as a function $Z\left(X', X''\right)$, we can rewrite EVE as $Z\left(X', E\left[X''\right]\right)$. Since Z is convex in X' (see Proposition 4.2 in Birge and Louveaux (2011) for a proof of the convexity of SSE), by Jensen's inequality, we have:

$$Z\left(X', X''\right) \geq Z\left(X', E\left[X''\right]\right), \tag{73}$$

that is:

$$SSE \geq EVE. \tag{74}$$

□

4. Value of the stochastic and semi-stochastic efficiency

According to Birge (1982), the *value of the stochastic solution* (VSS) indicates the gain from solving the recourse problem rather than its expected value counterpart, i.e., the problem in which the random parameters are replaced with their expected values. A large VSS means that uncertainty greatly affects the optimal solution and that solving the expected value model would not be relevant. In the same manner, we name the difference between the SE and the EVE as the *value of the stochastic efficiency* (VSE), where:

$$VSE = SE - EVE, \tag{75}$$

and the difference between the SSE and the EVE as the *value of the semi-stochastic efficiency* (VSSE), where:

$$VSSE = SSE - EVE. \tag{76}$$

A numerical illustration

Six experiments ($E1, \ldots, E6$) have been carried out with 10 DMUs ($j \in \{1, \ldots, 10\}$) including one controllable input X, one stochastic uncontrollable input $X(\omega)$, and a constant output Y. The output variable is equal to 100 across all of the DMUs. In a Monte Carlo sampling approach, two thousand scenarios have been generated for the uncontrollable input $X_j(\omega)$, which follows, for the sake of simplicity, a normal distribution with the mean 20, and the standard deviation defined as $s_j = 1 + 0.5j$ for the j^{th} DMU. It is to be noted that one can use any empirical distribution instead of normal distribution. We fixed the value of X_j to $10 + (j-1)\Delta$, for the first three experiments, and the value of X_{11-j} to $10 + (j-1)\Delta$ for the last three experiments, where $\Delta \in \{1, 2, 3\}$.

For all of the experiments, we computed SE, SSE, and EVE, and reported in Figures 1 to 4 the efficiency and the value of the objective function of each model ($P9$, $P8$, and $P10$). In Figure 1, the mean stochastic efficiency is decreasing with the Δ, and so does the consistency. For the DMUs 1 to 6, the stochastic efficiency is decreasing with the Δ. We observe that the phenomenon is reversed from DMU 7 onwards due to a higher variance of the uncontrollable input. As DMUs 9 and 10 have an even higher variance, they retain their positions on the frontier, irrespective of the Δ. This phenomenon is perfectly observed in experiments 4 to 6 in Figure 2, where the DMUs 6 to 10 are stochastically efficient. In Figures 1 to 4, we observe that propositions 1 and 2 hold. In the first three experiments (Figures 1 and 3), the variance of the stochastic input and the value of the deterministic input are both increasing from DMU 1 to DMU 10, and this reflects the respective increase in the value of the stochastic efficiency.

In the last three experiments (Figures 2 and 4), where the variance of the stochastic input increases from DMU 1 to DMU 10, and the value of the deterministic input is decreasing from DMU 1 to DMU 10, we observe that the VSE and $VSSE$ decrease with either the lower stochasticity or the lower controllable input among all of the DMUs.

5. Conclusions

A stochastic and a semi-stochastic DEA models are proposed and compared with the deterministic DEA model. We show that the difference between the SE or the SSE and the deterministic efficiency is non-negative, and so we define the VSE and the $VSSE$, respectively. It is worth noting that the proposed models do not call for any specific theoretical statistical distribution: they can be used with any empirical distribution.

This paper demonstrates that solving a stochastic DEA problem by using a DEA model with expected values of the random inputs could lead to wrong decisions. Indeed, the DMU of interest could be erroneously discarded from the efficient frontier when the stochastic information is either unobserved or limited to a measure of central tendency.

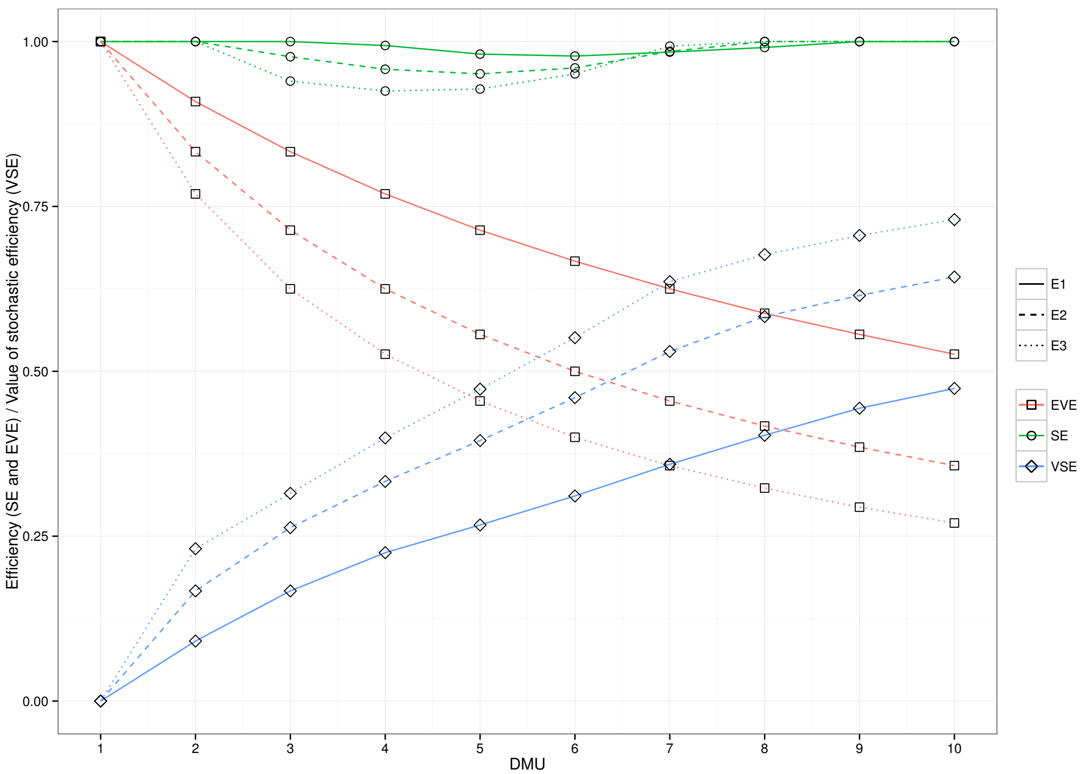

Figure 1: Stochastic efficiencies and Value of the stochastic efficiencies for E1, E2, and E3

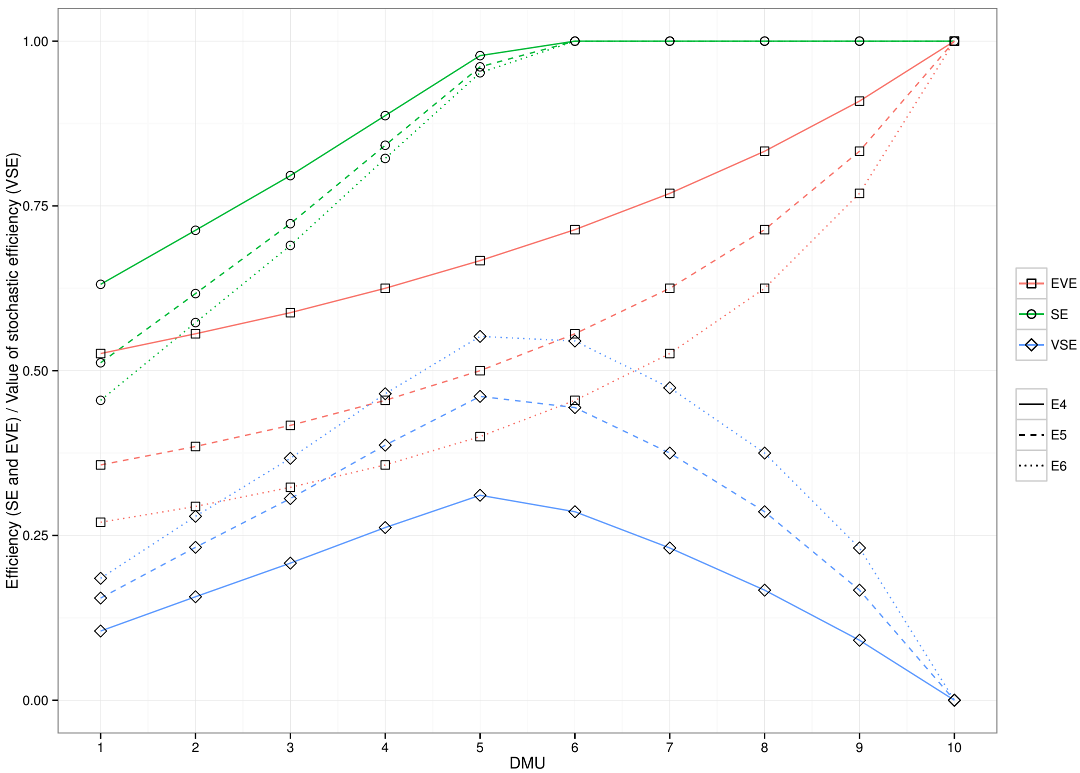

Figure 2: Stochastic efficiencies and Value of the stochastic efficiencies for E4, E5, and E6

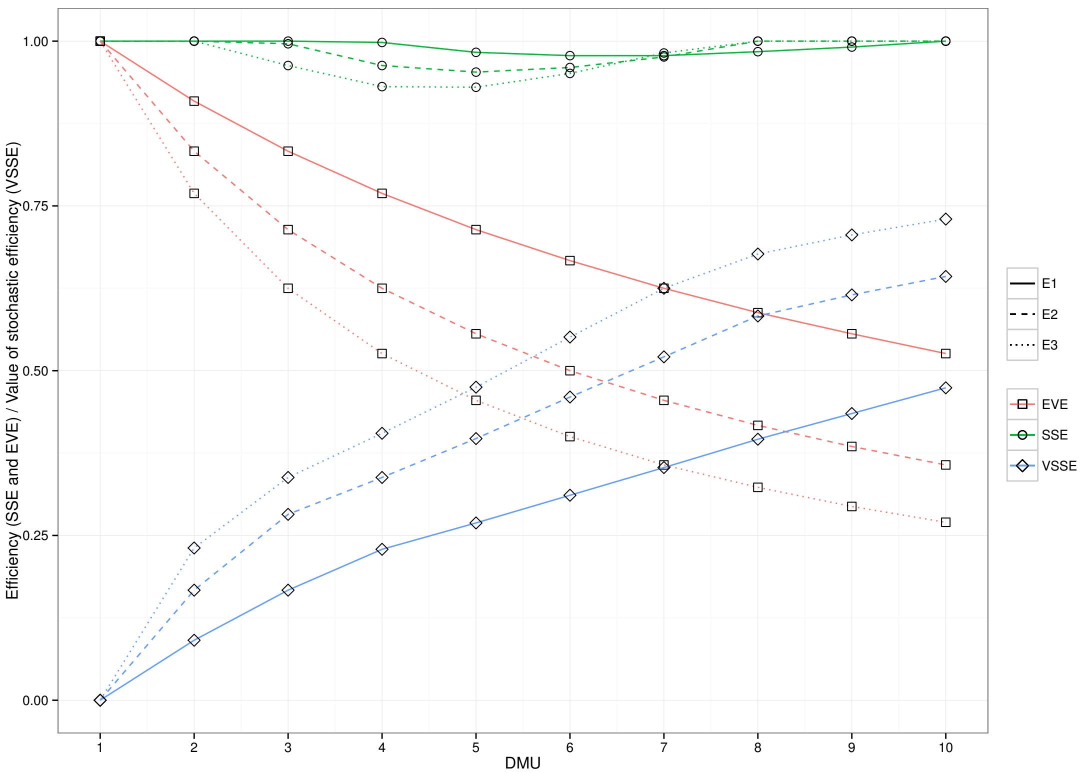

Figure 3: Semi-stochastic efficiencies and Value of the stochastic efficiencies for E1, E2, and E3

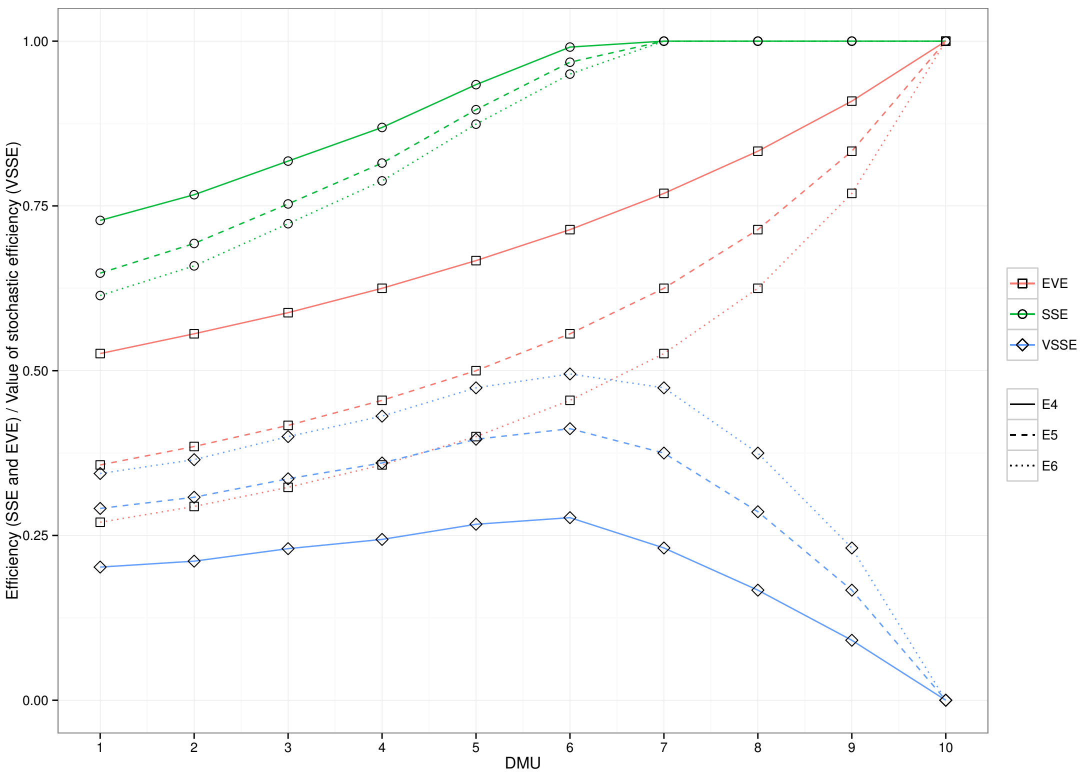

Figure 4: Semi-stochastic efficiencies and Value of the stochastic efficiencies for E4, E5, and E6

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